Chaotic dynamics of a body-vortex pair

Johan Roenby\textsuperscript{a,b}, Hassan Aref\textsuperscript{b,c}

\textsuperscript{a}Department of Mathematics, Technical University of Denmark, Kgs. Lyngby, Denmark
\textsuperscript{b}Center for Fluid Dynamics, Technical University of Denmark, Kgs. Lyngby, Denmark
\textsuperscript{c}Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, USA

Abstract
We study an idealized model of body-vortex interaction in two dimensions. The fluid is incompressible and inviscid and assumed to occupy the entire unbounded plane except for a simply connected region representing a rigid body. There may be a constant circulation around the body. The fluid also contains a finite number of point vortices of constant circulation but is otherwise irrotational. We assign a mass distribution to the body and let it move and rotate freely in response to the force and torque exerted by the fluid. Conversely, the fluid moves in response to the body motion. We study the occurrence of chaos in the system of ODEs emerging from these assumptions. It is well-known that the system consisting of a circular body with uniform mass distribution interacting with a single point vortex is integrable. Here we investigate how this integrability breaks down when the body center-of-mass is displaced from its geometrical center. We find two distinct regions of chaos and discuss how they relate to the topology of the trajectories of body and vortex.

Keywords: body-vortex interaction, point vortex, ideal flow, dynamical system, chaos, integrability

1. Introduction
The interaction of a bluff body and a few vortices is a recurring theme in many contexts, including the response of man-made structures to an oncoming flow, the swimming of fish, and the flight of insects. We have been exploring a class of models of body-vortex interactions that are, on one hand, extremely simple – some may say oversimplified – yet, on the other hand, surprisingly rich in the variety of dynamics they can display. Our models are based on the following premises: (1) The fluid is ideal, i.e., incompressible and inviscid; (2) The fluid motion is two-dimensional and unbounded externally. Except for a finite number of point vortices embedded in it, the fluid is in potential flow; (3) The body is a rigid body of arbitrary two-dimensional shape, although simply-connected and of finite extent. It may have a non-uniform mass distribution. It is free to move in the fluid in response to forces and torques exerted on it by the fluid. As ideal flow theory allows, it may have a prescribed constant circulation around it; (4) All vortices present in the flow are represented as point vortices of constant circulation.

The main theoretical advantage of this idealization is that the complex system of coupled PDEs for the fluid motion and ODEs for the body motion reduces to a system of ODEs for the motion of the body and however many
point vortices are introduced. This reduction is exact modulo the singularity inherent in the point vortex approximation and corresponds to some kind of “weak solution” to the Euler equations with embedded vorticity. The point vortex approximation has a long and rich history as a modeling tool in this sense in fluid mechanics, that we shall not re-iterate here.

The advantages of working with a system of ODEs are several. The main one, from our perspective, is that the tools of dynamical systems theory become available. These provide a new language of inquiry for many issues in body-vortex interactions. Apparently random fluctuations in forces and torques, and sensitive dependence on initial conditions now have a natural context in the sense of chaos in a dynamical system.

One can view our work as extending the modeling of vortex wakes by rows of point vortices, introduced by von Kármán a century ago, to body-vortex interactions where the body is free to move and the shed vortices are introduced “by hand” to mimic the situation of interest. For example, just after a vortex has been shed from a body, the circulation around the body is opposite to the circulation of the vortex. To model this situation, then, we would impose a circulation around the body that is opposite to the circulation of the vortex. The actual process of vortex shedding cannot, of course, be captured by this kind of modeling.

The equations of motion of body and vortices arise by combining several classical theoretical developments: For the vortices we use the Kirchhoff-Routh-Lin theory of point vortex motion in a domain with rigid boundaries on which we apply ideal flow boundary conditions (Lin, 1943). For the body we exploit the invariance of the dynamics to rigid displacements and rotations of the body-vortex system. This leads to conservation of the body linear momentum plus fluid linear impulse, \( P = P_x + iP_y \), in complex notation, and body angular momentum plus fluid angular impulse, \( L \). In effect, these are integrals of the full system that would result from combining Kirchhoff-Routh-Lin theory with the Kirchhoff-Kelvin equations for the motion of the body supplemented by expressions for the force and torque on the body produced by the vortices. The equations for \( P \) and \( L \) in terms of the dynamical variables may be inverted to obtain expressions for the instantaneous body linear and angular velocity from the instantaneous body-vortex configuration. The resulting equations of motion are quite complex and have only been fully derived in recent years (Shashikanth, 2005; Borisov et al., 2007; Roenby & Aref, 2010). In most cases they are beyond the realm of analytical solution and require numerical calculation. The combination of the complexity of the equations and the necessity of numerical tools has apparently delayed the elucidation of the effects presented here for several decades.

Due to space constraints on this article, we will not repeat the equations of motion but simply refer the reader to Roenby & Aref (2010), henceforth referred to as RA, which gives a clear and concise presentation. Unfortunately, there are inaccuracies in some of the statements of the equations in the literature, which are only revealed when one starts calculating things numerically and comparing to analytical expectations in various limits (e.g., when the rigid body is shrunk down to a point vortex). For a detailed derivation of the equations of motion and discussion of previous work the reader is referred to Roenby (2011).
2. Mechanisms of chaos

In RA we presented a numerical exploration of the transition to chaos for the case of a single vortex interacting with an elliptic body with uniform mass distribution and with opposite circulation of the vortex around it. This transition was first demonstrated in Borisov et al. (2007) although with almost no explanation of what was being shown in the diagrams presented. By numerically calculating body and vortex trajectories and constructing Poincaré sections from these, we discovered two chaotic regimes for the system.

The first chaotic region originates from the integrable system of a uniform circle interacting with a point vortex (Borisov & Mamaev, 2003). When \( P \neq 0 \) the circle-vortex pair drifts in the direction of \( P \). Depending on the initial conditions the body and vortex either orbit one another or travel side-by-side while steadily drifting in the direction of \( P \). Between these two modes there is an unstable relative equilibrium connecting two homoclinic orbits. Adding “noise” to a vortex on one of these orbits, would make it shift randomly between the different modes. This is exactly what happens when the body is made slightly elliptic: No randomness is introduced, but the coupling with the body rotation now effectively serves as a chaos-inducing perturbation.

The second chaotic regime found in RA we proposed to understand in a similar manner by considering the integrable motion of a body with no circulation or vortices around it moving freely in an unbounded, ideal fluid. There are three steady motions of the body, namely two purely translational modes and one purely rotational mode, the latter corresponding to vanishing \( P \) (Lamb, 1932). The two purely translational “eigen-modes” are in perpendicular directions with one being stable and the other being unstable to small perturbations. If we add “noise” to the unstable eigen-mode, the body will start rotating in a random manner while propagating through the fluid in the direction of \( P \). A similar effect can be obtained by adding a coupling with a point vortex in the fluid which will then act as a chaotic-motion-inducing force on the body. The rotational motion of the body then generates a velocity field in the fluid that pushes back on the point vortex.

In the present paper we explore a similar case, still with just one point vortex, but now with a circular body with center-of-mass (slightly) displaced from its geometrical center. Certainly most real-world rigid bodies, man-made or natural, have nonuniform mass distribution with their center of mass displaced from their geometrical center. It is not difficult to prove that a freely moving circle with no vortices around it will have constant angular velocity even in the case of nonuniform body mass. This is therefore a degenerate case where the body has no unstable translational eigen-modes. Thus, if the second chaotic region discussed in RA does indeed have its origin at such an unstable mode, we should not see this region in the circle-vortex system studied here where the circle is nonuniform.

3. Poincaré sections

Without loss of generality we may choose our laboratory coordinates such that the position of the geometrical center of the body at time \( t = 0 \) is at the origin, i.e such that \( z_0(0) = 0 \) in complex notation. The body orientation, \( \theta \), is the angle between the \( x \)-axis of the laboratory frame and the abscissa of a coordinate system fixed in the body and
centered at $z_0$. We may choose the laboratory coordinates and the body-fixed coordinates to be initially aligned such that $\theta(0) = 0$. We choose our unit of length such that the radius of the body is 1, and our unit of mass such that the fluid density is 1. The remaining parameters are the vortex strength, $\Gamma_v$ – the circulation around the body is assumed to be $-\Gamma_v$, corresponding to the case of a shed vortex – the body mass, $m$, the location of the center of mass, $z_{cm}$, and the moment of inertia, $I_0$ (with respect to $z_0$). We choose our unit of time such that $\Gamma_v = 1$. We assume the body to be “neutrally buoyant” (although there is no gravity in the formulation), i.e. $m = \pi$, and its center of mass to be initially placed at $z_{cm}(0) = 0.1$. Clearly there are many ways to distribute mass inside the body such that it has mass $m$ and center of mass at $z_{cm}(0)$. These in general result in different values for the moment of inertia $I_0$. It would be interesting to study the effect of different mass distributions giving rise to different values of $I_0$, but here we simply choose to consider $I_0 = \pi/2$. This is the same value as for a uniform circle of mass $m$. The choices for $m$, $z_{cm}(0)$ and $I_0$ also encompass a variety of possible mass distributions of the body. Insofar as the body-vortex dynamics to be studied is concerned, it is only these three moments of the mass distribution that are of importance.

Having chosen values for the model parameters we must now specify the remaining initial conditions. These are the initial vortex position, $z_v(0)$, the initial velocity of the body, $\dot{z}_0(0)$, and its initial angular velocity, $\dot{\theta}(0)$. The parameters and initial conditions uniquely determine the values of the four conserved quantities of the system dynamics: $P$, $L$ and the fluid plus body kinetic energy which we denote $H$. Instead of specifying $(z_v(0), \dot{z}_0(0), \dot{\theta}(0))$ we
Figure 2: Poincaré sections generated from numerical solutions for all initial conditions on the $H = 1$ contour in Fig. 1 where there is an arrow. A point is plotted each time the body orientation $\theta$ (mod $2\pi$) returns to zero. In the left panel the reduced phase space in which the points are drawn is spanned by position coordinates of the vortex and body. In the right panel it is spanned by the body linear and angular velocities. Points for which $\dot{\theta} > 0$ ($\dot{\theta} < 0$) are colored red (black).

could specify $(z_v(0), P, L)$ since $\dot{z}_v(t)$ and $\dot{\theta}(t)$ at any time $t$ are uniquely determined by the body-vortex configuration at that time and the parameters $P$ and $L$. Here we choose (somewhat arbitrarily) to consider initial conditions with $(P, L) = (2 + 0i, -2.5)$. Having specified $P$ and $L$ and the body initial position and orientation, the energy $H$ is now only a function of the initial vortex position $z_v(0) = x_v(0) + iy_v(0)$, and we may plot contours of $H(x_v(0), y_v(0))$. The three black curves in Fig. 1 are such contours of initial vortex positions having the same values of $(P, L, H)$. The initial linear and angular body velocity required to keep $(P, L)$ constant varies along such a curve. For different points along the $H = 1$ contour the required body initial velocity, $\dot{z}_v(0)$, is shown by blue arrows. Also three contours of the required initial angular body velocity, $\dot{\theta}(0)$, are shown as red, dashed curves. The diagram in Fig. 1 then, summarizes how to choose initial data in order to follow body-vortex motion for fixed parameters and chosen values of the integrals of motion.

In order to investigate variations in the behavior of the system for fixed values of the integrals of motion we have obtained numerical solutions for each of the approximately 50 initial vortex positions along the $H = 1$ contour in Fig. 1 with a $\dot{z}_v(0)$-arrow attached to it. We are particularly interested in the occurrence of chaos in the system and so generated Poincaré sections in the following way: Each time the body orientation (mod $2\pi$) returns to its initial value $\theta(0) = 0$ we plot a point in a three-dimensional reduced phase space consisting of two components of the vortex position relative to the body position, $x_v - x_0$ and $y_v - y_0$, along the first two axes, and the component of the body
position perpendicular to $P$, i.e., with our chosen value of $P$ just $y_0$, along the third axis. The result of the calculation is shown in the left panel of Fig. 2.

A consequence of choosing all the initial conditions to have the same values of $(P, L, H)$ is that the points in the Poincaré section all lie on a two-dimensional surface embedded in the three-dimensional reduced phase space. Most of the calculated solutions are quite regular resulting in the dots of the Poincaré section lining up on curves on this surface. This is maybe not so surprising considering the small perturbation from the integrable system of a vortex and a uniform circle that we have introduced by setting $z_{cm}(0) = 0.1$. Nevertheless there are some “fuzzy regions” in the section where the points from a single solution seem to occupy an area rather than a curve on the surface. To get a clearer view we plot in the right panel of Fig. 2 the points in another reduced phase space where the components of the body velocity are along the two first axes and the body angular velocity is along the third. From this we see that we may conveniently construct a two-dimensional Poincaré section by plotting points with $\dot{\theta} > 0$ and $\dot{\theta} < 0$ in different figures. In the left (right) panel of Fig. 3 we show the vortex position relative to the body each time $\theta = 0$ (mod $2\pi$) and $\dot{\theta} > 0$ ($\dot{\theta} < 0$).

4. Chaotic regions

From the projections presented in Figs. 2 and 3 it is clear that there are two distinct chaotic regions surrounded by regular regions on the surface seen in the Poincaré sections. The solutions marked by $\triangle$ and $\triangledown$ in the right panel
Figure 4: Body (red) and vortex (blue) trajectories and corresponding evolution of the body angular velocity, $\dot{\theta}$, for the first 300 time units of the solutions with the initial vortex position marked by $\triangle$ and $\nabla$ in the right panel of Fig. 3. The body-vortex pair travels from left to right. Their final positions are shown with a (red) dot and a circle, respectively, and also the final body orientation is displayed in the body. A black dot is placed for every 50 time unit.

of Fig. 3 are started with almost identical initial conditions. Nevertheless, the $\triangle$ solution exhibits regularity while the $\nabla$ solution forms part of the chaotic sea. To examine how this transition to chaos manifests itself in the body-vortex interactions we plot in Fig. 4 the vortex and body trajectories for the first 300 time units of the numerical solution used to generate the Poincaré section. We also plot the corresponding time evolution of $\dot{\theta}$. For the regular $\triangle$ solution the body-vortex pair travels side-by-side towards the right and the periodicity of both the trajectories and the $\dot{\theta}$ evolution is apparent. For the chaotic $\nabla$ solution, on the other hand, the body-vortex pair seems to be unable to “decide” whether to travel towards the right in a side-by-side mode or in the orbiting mode described in Section 2. The evolution of $\dot{\theta}$ has also become irregular for the $\nabla$ solution.

In a similar manner we examine trajectories at the edge of the other chaotic region appearing in the Poincaré sections. Here the $\triangleright$ solution in the right panel of Fig. 3 is regular while the $\triangleleft$ solution is chaotic. From Fig. 5 we see that this transition to chaos is not associated with a change in the orbit topology as in Fig. 4. The body-vortex pair moves towards the right in the orbiting mode in both panels of Fig. 5 though the motion becomes somewhat “wobbly” in the chaotic solution (right panel). For the evolution of $\dot{\theta}$ on the other hand there is a dramatic qualitative change between the two almost identical initial conditions.

Thus, while the first transition to chaos visualized in Fig. 4 has significant effects on both the fluid and the body motion, the second transition visualized in Fig. 5 is presumably mostly felt by the body and to a lesser degree affects
5. Concluding remarks

We have demonstrated the existence of two different regions of chaos for the interaction of a point vortex of constant circulation and a slightly inhomogeneous circular body of opposite circulation. The results are similar to those presented in RA where the body is a homogenous ellipse. In both cases we have made small parametric perturbations that break the rotational invariance. It is this invariance with respect to $\theta$ that causes the interaction of a vortex and a uniform, circular body to be integrable. Perturbing away from this integrable system by making the body slightly elliptic alters all terms in both the body and vortex equations of motion and introduces new terms, e.g., due to the non-zero flow now associated with rotation of the body around its geometrical center. By making the circle slightly inhomogeneous the vortex equations of motion are unaltered and the only change is the form of the effective mass tensor appearing in the body equations of motion. The effective mass tensor may be written

$$
\begin{bmatrix}
A & 0 & 0 \\
0 & A & \varepsilon \\
0 & \varepsilon & B
\end{bmatrix}
$$

where $A, B$ and $\varepsilon$ are constants. The only thing revealing the broken $\theta$ symmetry for the inhomogeneous circle is the displaced center of mass causing nonzero $\varepsilon$ in this tensor. In this sense making the body inhomogeneous is a “cleaner” perturbation of the integrable circle-vortex system than the one studied in RA. The fact that the second chaotic region (Fig. 5) also exists when the body is an inhomogeneous circle does not support our previous explanation of this region.
having to do with an unstable body eigen-mode. Thus, further investigations need to be made to obtain a mechanistic understanding of this chaotic regime. Such investigations might also reveal why the $\dot{\theta} > 0$ solutions in the left panel of Fig.3 exhibit much less sensitivity to the perturbation than the $\dot{\theta} < 0$ solutions in the right panel.

Admittedly, the price of simplifying the body-vortex interaction to the extent done here is that it hardly resembles a real-world fluid-structure interaction situation. Nevertheless, it is conceivable that the transitions between chaotic and regular behavior often observed in real fluid-structure interactions may originate from mechanisms similar to those identified here for the model system. Hence, in spite of its questionable reality, the simplified model may serve as an important tool for obtaining insights into certain aspects of real fluid-structure interactions. It would certainly be of interest to see whether mechanisms for onset of chaos such as those identified here could be recognized in experiments or in CFD simulations.

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