Nonlinear excursions of particles in ideal 2D flows

H. Aref\textsuperscript{a,d,*}, J. Roenby\textsuperscript{b,d}, M.A. Stremler\textsuperscript{a}, L. Tophøj\textsuperscript{c,d}

\textsuperscript{a} Department of Engineering Science & Mechanics, Virginia Tech, Blacksburg, VA 24061, USA
\textsuperscript{b} Department of Mathematics, Technical University of Denmark, Lyngby, DK-2800, Denmark
\textsuperscript{c} Department of Physics, Technical University of Denmark, Lyngby, DK-2800, Denmark
\textsuperscript{d} Center for Fluid Dynamics, Technical University of Denmark, Lyngby, DK-2800, Denmark

\textbf{ARTICLE INFO}

Article history:
Available online xxxx

Keywords:
Particle motion
Ideal flow
Point vortex
Rigidbody
Chaos

\textbf{ABSTRACT}

A number of problems related to particle trajectories in ideal 2D flows are discussed. Both regular particle paths, corresponding to integrable dynamics, and irregular or chaotic paths may arise. Examples of both types are shown. Sometimes, in the same flow, certain particles will follow regular paths while others follow irregular paths. Even in the chaotic region the amount of regularity or irregularity of a path depends on initial conditions and system parameters. The notion of a transported fluid region or “atmosphere” is mentioned. Various conclusions, ideas and queries are formulated based on the examples given. The intimate mix of regular and chaotic trajectories complicates a purely Lagrangian approach to fluid flow problems.

1. Introduction

It is well known that fluid dynamical problems may be approached using either the Lagrangian or the Eulerian representation. In a number of applications, fluid stirring and mixing problems in particular, the Lagrangian representation captures important information on the motion and distribution of individual particles. The appearance of chaotic particle trajectories, which is possible even in very simple flows, is an important phenomenon, both fundamentally and from the perspective of applications. We shall refer to the complex motions that passive particles, vortices, or rigid bodies interacting with their ambient flow undergo as nonlinear excursions. In this paper we summarize some of our experiences in this problem area by juxtaposing various examples. We work in an idealized “universe” that consists of 2D ideal flow, point vortices, freely moving rigid bodies (with slip boundary conditions) and passively advected particles. We survey a number of cases and formulate some conclusions that we hope may be of interest and applicability beyond our idealized flow situations.

It is a pleasure to dedicate this paper to Lou Howard on the occasion of his 80th birthday.

2. Advection

The simplest problem of the type under consideration is the motion of a passively advected particle in a prescribed flow.

Equivalently, this problem addresses the kinematics of the flow itself since every particle of the fluid may be thought of as being passively advected by the flow of which it is a part. The potential complexity of this problem is immediately apparent by considering the case of 2D incompressible flow given by a streamfunction \( \psi(x, y, t) \):

\[
\frac{dx}{dt} = \frac{\partial \psi}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial \psi}{\partial x}. \tag{2.1}
\]

These equations are in the form of Hamilton’s canonical equations for a one-degree-of-freedom system, with \( x \) and \( y \) as the conjugate variables and with the stream function, \( \psi(x, y, t) \), playing the role of Hamiltonian. This identification derives from the incompressibility of the flow, and so pertains both to inviscid and viscous flows. The dissipative dynamics of a viscous fluid can still have “conservative” Hamiltonian kinematics so long as the fluid is incompressible.

A pair of conjugate variables in a Hamiltonian system are usually thought of as a “generalized coordinate” and a “generalized momentum”. In the case of flow kinematics or advection both conjugate variables are coordinates of the advected particle. In this sense, as advection by an incompressible 2D flow is followed in real space, one is peering into the phase space of the Hamiltonian system defined by its flow kinematics. While this identification must have been known for quite some time, its consequences in terms of fluid stirring and mixing were realized only more recently\[1]\.

If \( \psi \) is independent of time, Eqs. (2.1) constitute two first-order ODEs with a conserved quantity, namely \( \psi \) itself. Such a problem is
clearly integrable. More formally, the rate of change of $\psi$ following a particle is

$$\frac{D\psi}{Dt} = \frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial t} .$$

Hence, $\psi$ is a Lagrangian invariant precisely when it has no explicit time dependence, i.e., precisely when the flow is steady. For time-dependent $\psi$, we have a non-autonomous Hamiltonian system with one degree of freedom, sometimes said to have “one-and-a-half degrees of freedom”. Such a system is closer in its properties to a two-degree-of-freedom system than to the autonomous, steady flow, one-degree-of-freedom system.

As Arnold [2] put it, somewhat provocatively, in his textbook: “Analyzing a general potential system with two degrees of freedom is beyond the capability of modern science”. In other words, advection by a 2D, unsteady, incompressible flow is a surprisingly difficult problem.

Chaotic advection by a 2D, unsteady, incompressible flow is a topic in the dynamical system (2.1) leads to nonlinear excursions where particles initially placed close to one another may deviate exponentially. In turn, this chaotic advection leads to efficient stirring of the fluid and, in due course, to enhanced mixing. See [3] for a historical perspective on the development of this idea. Applications of chaotic advection, sometimes called chaotic mixing, encompass length scales from thousands of kilometers (in geophysical applications) to microns (microfluidics).

Two classical examples related to particle advection are worth noting. The first is Kelvin’s idea of the “atmosphere” of a vortex pair [4]. In the problem of two point vortices (see Section 3), a pair of opposite vortices will translate side by side, the 2D counterpart of a vortex ring. Kelvin noticed that if one goes to the rest frame of the pair, the steady streamline pattern has an oval envelope surrounding the vortices. Fluid within the envelope will advect forward with the pair, while fluid exterior to the envelope travels around the oval dividing streamline and is eventually left behind. The original diagram (with the horizontal streamline added) is reproduced in Fig. 1(a). A trapped region of fluid of this type is usually referred to as the atmosphere of the vortex pair.

The appearance of a region of fluid in a bound motion about a point vortex is generic due to the vortex being an isolated point about which there is circulation in an otherwise irrotational flow. Fig. 1(b) shows a more elaborate situation with three vortices having zero net circulation placed at the vertices of an equilateral triangle [5]. This configuration is a relative equilibrium that translates uniformly. The gross features of the streamline pattern in the rest frame of such a translating vortex triple are similar to the streamline pattern around the translating dipole in Fig. 1(a). In particular, the velocity field far from the tripe is similar, to leading order, to the flow field far from the dipole. There is a forward and a rearward stagnation point, and there are regions of fluid that orbit each of the three vortices and are carried along with them. However, the specific pattern of the dividing streamlines for the vortex tripe is considerably richer than for the vortex dipole.

The single streamline that impinges on the vortex pair atmosphere from the front and departs from the rear has been broadened into a band of fluid that enters the tripe at its front, loops around inside it, and exits from the back. There is also a region of fluid that orbits both of the two positive vortices.

There are simpler examples as well: for a cylinder of small radius with circulation around it, a common model of a spinning cylinder in a uniform stream of viscous fluid, there will be a trapped region around the cylinder when its angular velocity is high enough that the stagnation streamline detaches. This example is usually included in introductory texts as part of the discussion of lift on a spinning cylinder. The region inside the detached, self-intersecting streamline constitutes an atmosphere that follows the cylinder.

Since streamlines and pathlines coincide for steady flows, these examples illustrate that questions of particle trapping around vortex configurations or spinning bodies can be decided quite easily in cases where the fluid motion becomes steady in a suitably chosen rest frame.

In an appropriate frame of reference, a point where a (steady) streamline self-intersects is a stagnation point of the flow. For an incompressible flow, such a point will generally be a saddle point. The trapped region, if there is one, will then have one boundary that is a homoclinic or heteroclinic saddle connection of the streamline pattern. This leads to the trapped region being, in general, unstable to perturbations. Thus, if a steady flow with this kind of flow feature is made unsteady by a perturbation, the trapped fluid will tend to "leak out" due to the disintegration of the saddle point into a homoclinic or heteroclinic tangle. This is a general mechanism of chaos. Due to the leakage, the question of long-term existence of trapped regions in unsteady flow becomes quite difficult and requires advanced tools from the theory of dynamical systems. For background and details see any of several texts on chaotic dynamics, e.g., [8]. For a collection of articles that deal with the so-called "Lagrangian coherent structures", essentially contours of Lyapunov exponents, see [9].

All the stagnation points shown in Fig. 1, except for the vortices themselves, are hyperbolic. Leakage from a configuration such as that in Fig. 1(a) due to an externally applied perturbation was analyzed in detail by Rom-Kedar et al. [10]. Applying their results to a configuration such as Fig. 1(b), we see that the hyperbolic point located between the two positive vortices is not "dangerous"—even if it leaks due to perturbation, the trapped fluid will still remain inside a confined region. (One can imagine sufficiently localized perturbations that the inner saddle connection is destroyed, but the outer one remains intact.) The front and rearward hyperbolic points, however, will leak to the ambient fluid exactly as in the case of the vortex pair. Indeed, in this case leakage may lead to an enhancement of the transport through the dipole that is already present in the steady state. A systematic analysis of this case awaits execution.

The second example we highlight is the analysis by Maxwell [6]. The advection due to uniform translation of a cylinder through
2D ideal fluid otherwise at rest is considered. Maxwell showed that the advection is integrable in this case by directly integrating the resulting advection Eqs. (2.1) in the rest frame of the cylinder where the flow is steady. He produced plots of both the deformation of material lines and the trajectories of individual particles. These are reproduced in Fig. 2. The characteristic looping motion, where the particle is first pushed ahead and away from the cylinder, and subsequently is sucked in behind it as it moves past, will reappear in what follows. While particles that are initially close to one another will certainly deviate as the flow progresses, they will only do so algebraically, not exponentially as in the case of chaotic motion. Morton [11] extended Maxwell’s analysis to elliptical cylinders and also considered the case of uniform rotation. We shall return to recent extensions of this work in Section 4.

With one notable exception, interest in Lagrangian aspects of fluid flows, including particle trajectories, seems to have waned for several decades in the first half of the 20th century. The paper by Darwin [12], which introduced the notion of drift, was published some 40 years after Morton’s study (and does not cite his work). Darwin’s work was famously taken up a few years later by Lighthill [13] in a paper that probably can claim to have the shortest title of any in fluid mechanics. The notable exception is the study of vortex motion, a subject where Lagrangian considerations play a very significant role. Tracking vortices analytically, numerically or experimentally has been of considerable interest for a long time. We turn to this topic next.

3. Point vortices

These intriguing singular structures and their equations of motion were introduced by Helmholtz [14] in his seminal paper. On the unbounded plane, thought of as the complex plane, we consider $N$ point vortices of constant circulations $\Gamma_1, \ldots, \Gamma_N$. The positions of the vortices at time $t$ may be thought of as complex numbers $z_1(t), \ldots, z_N(t)$. If we only have the vortices, and no boundaries or imposed potential flow, the mutually induced motion is given by the set of ODEs:

$$\frac{dz_\alpha}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^{N} \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \quad \alpha = 1, \ldots, N. \tag{3.1}$$

The bar over the vortex velocity on the left-hand side indicates complex conjugation. See [15] or any of several texts on fluid mechanics for a derivation and discussion. The earliest use of the complex variable representation is unclear, but it was certainly used by Friedrichs [16] in his lectures on the subject. The prime on the sum on the right-hand side tells us to omit the singular term $\beta = \alpha$.

The solution to the two-vortex problem was given by Helmholtz. The solution of the three-vortex problem for arbitrary circulations is due to Gröbli [17], see [15,18] for further background. The trajectories for two-vortex motion are shown in Fig. 3. Two vortices generally orbit on concentric circles. They move opposite to one another when their circulations are of the same sign, Fig. 3(a), and together when they are of opposite sign, Fig. 3(b). In the special case...
case when their circulations are exactly opposite, they translate along parallel lines with uniform velocity. The atmosphere for a translating vortex pair, Fig. 1(a), is the streamline pattern in the co-moving frame of the motion depicted in Fig. 3(c).

The trajectories for the three-vortex problem vary with the vortex circulations and, for given circulations, with the initial positions of the vortices. Fig. 4 shows a series of trajectories for three identical vortices as the initial positions are varied. In the initial panel, Fig. 4(a), two of the vortices are much closer together than either one is to the third vortex. In effect, these two form a bound state that acts much like a single vortex of twice the strength. This “compound vortex” then orbits the third vortex, as we see in Fig. 3(a), although its internal structure leads to a small periodic perturbation of the overall motion. As we make the three inter-vortex distances more and more similar in size, we produce more and more “collective motion” in which there are no bound states between any two of the vortices. Ultimately, as we approach an initial condition in the shape of an equilateral triangle, where the three vortex separations are equal, we obtain a relative equilibrium which, for three identical vortices, is stable. The regularity of the trajectory plots in Fig. 4 arises from the motion being quasi-periodic as one would expect in an integrable system.

Fig. 5 shows something quite different. We are again looking at vortex trajectories, this time during the collision and interaction of two vortex pairs, i.e., a four-vortex problem. The vortices have circulations ±1 in each pair. This problem was explored by Price [19] and more recently by Tophøj and Aref [20]. For the motion shown the vortices colored blue and red are positive, the other two negative. The two pairs have been set on a collision course, with the two panels in Fig. 5 tracing the evolution from imperceptibly different initial conditions. A complicated collision interaction process takes place with numerous nonlinear excursions of the vortices until, after some time, two pairs re-emerge. The initial four loops are similar in the two panels, but then the motions depart quite radically from one another since the motion is chaotic. Indeed, in Fig. 5(a) the two original pairs re-emerge in a process we call direct scattering. In Fig. 5(b), on the other hand, the vortices switch partners in a process we call exchange scattering.

Four-vortex problems are generally non-integrable, although for special choices of the circulations and the initial conditions we may have integrability. The most interesting example is for the case $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 = 0$ and

$$\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 + \Gamma_4 z_4 = 0.$$  \hspace{1cm} (3.2)

For choices of the four circulations and initial positions that satisfy these two constraints the problem of relative motion of the vortices can be “projected down” to a three-particle problem, akin to the three-vortex problem, and a complete analysis performed [21,22]. When the motion is bounded, the relative motion is periodic, and trajectories of the four vortices are reminiscent of the trajectories seen in Fig. 4. For unbounded motions, we obtain regular counterparts to the scattering trajectories in Fig. 5. Aref and Strelmer [22] give many examples of trajectory plots.

In the limit when one of the four vortices has circulation zero and so has become a passive particle in the flow field generated by the three remaining vortices, we have an interesting split between the four fluid particles under consideration: the three vortices move integrably as in Fig. 4, while the passively advected particle moves chaotically according to the considerations in Section 2. This restricted four-vortex problem [23] was the precursor to the general idea of chaotic advection. As an amusing historical aside we may mention that in his thesis [17], which was the first publication in which the three-vortex problem was solved, Gröbli wrote: “Auf die Bestimmung der Bewegung von Flüssigkeitstheilchen welche sich in endlicher Entfernung von den Wirbelfäden befinden, werden wir nicht eingehen”. [English translation: “We shall not enter into the determination of the motion of fluid particles located at a finite distance from the vortices.”] This suggests that he had tried the problem of advection by three vortices and found it impenetrable to analysis. Today we understand the reason: one cannot make much analytic progress with a chaotic system.

Fig. 6 provides an example of restricted four-vortex motion in which the three vortices move regularly and the passive particle moves irregularly. Not only is the trajectory of the advected particle clearly less regular than the vortex motion, the details of the trajectory are also sensitive to small perturbations of the initial particle position. Between panels (a) and (b) in Fig. 6, the initial position of the advected particle has been shifted by 0.0025 times the initial separation between vortices 1 and 2.

When might advection by three vortices be integrable? If we think of the problem as the limiting case of the four-vortex problem with $\Gamma_4 = 0$, then the case of advection by three vortices with $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ and $\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 = 0$ should be integrable. Indeed it is but, unfortunately, in a trivial way: for vanishing linear impulse the solution to the problem of three vortices with net circulation zero is a rigidly rotating, collinear tripoole [24,25]. For the symmetric case, $\Gamma'_1 = \Gamma'_2 = \Gamma'_3$, it is related to the well-known tripole vortex of geophysical fluid dynamics [26]. The advection about such a structure is clearly integrable as one sees by transforming to the co-rotating frame of the tripole.

There are other instances of integrable advection by three vortices (e.g., advection by a translating tripoole as in Fig. 1(b)) but, in general, the advection is irregular, and the restricted four-vortex problem is chaotic.

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Fig. 4. Vortex trajectories for motion of three identical vortices in 2D ideal flow. In (a) the distance between two of the vortices is much smaller than the distances of these two from the third vortex. As we progress through (b)–(f), the initial positions are changed such that the distances between the three vortices become more and more alike.
Fig. 5. Vortex trajectories for the collision-interaction of two vortex pairs. Each vortex is color-coded, with blue and red vortices having strength $+1$, green and purple vortices having strength $-1$. Initial conditions vary imperceptibly between (a) and (b), yet the long-term outcome is very different since the interaction takes place in a chaotic regime.

Fig. 6. Adveected particle trajectory (black) in the field of three identical vortices in 2D ideal flow (trajectories shown as grey). The vortex motion is of the integrable variety seen in Fig. 4. The trajectory of the advected particle is clearly much less regular than the vortex trajectories. Furthermore, small perturbations of the starting position of the particle (shown by black dots in the two panels) lead to substantial deviations in the trajectory.

4. Rigid bodies

We may also consider particle motion problems that include, in addition to point vortices, a rigid body that is free to move and, inevitably, advected particles. The motion of the body, which is launched with a linear and angular velocity, is affected by the reactive pressure forces from the fluid, which may include contributions from one or more point vortices. Apart from the motion of the body, and the choice of initial positions of the vortices, the fluid is assumed “otherwise at rest”. We now enter the topic of flow-structure interactions viewed from a Lagrangian perspective.

Incorporating a rigid body into the equations of motion requires several adaptations: first, the moving rigid body produces a potential flow that must be added to the right-hand sides of Eq. (3.1). The calculation of the potential flow induced by the body motion is, in essence, classical and one can find an algorithm for it in the text by Milne-Thomson [27]. Since the fluid domain outside a finite rigid body is no longer simply connected, ideal flow theory also provides the freedom to assign a constant circulation around the body. Such a circulation, in turn, adds a term to the potential flow. Next, the mere presence of the rigid body changes the boundary conditions on the vortex flow and thus the motion of the vortices: the vortices have “images” inside the rigid body. This part of the modification to Eq. (3.1) is essentially the Kirchhoff–Routh–Lin theory of point vortex motion outside a rigid boundary [28]. The third and final ingredient required is the equations of motion of the body itself. These are, in essence,
the Kirchhoff–Kelvin equations for rigid body motion, now with the addition of the forces due to the vortices and the circulation around the body. A convenient way of stating these equations in the present case turns out to be conservation of linear and angular impulse for the entire system, since no external forces act. These equations are available at several places in the literature [29–33]. The formal statement of the equations of motion is cumbersome and so we omit it here.

Although the analysis covers general body shapes and arbitrary mass distributions within the body, most studies have been done for ellipses with the further specialization that the ellipse has a homogeneous mass distribution within it, so the geometrical center of the ellipse coincides with its center of mass. Ellipses are given by the Riemann mapping

\[ f(\zeta) = \zeta + \frac{a^2}{\zeta}, \]  

(4.1)

from a unit circle in a second, complex \( \zeta \)-plane to the physical plane. For \( 0 \leq a \leq 1 \), the mapping (4.1) sends \( \zeta = e^{i\theta}, 0 \leq \theta < 2\pi \), to an ellipse in the physical plane with foci at \( \pm 2a \) and eccentricity \( \frac{2a}{1+a^2} \). The case of a circular cylinder corresponds to \( a = 0 \). For \( a = 1 \), we obtain a flat plate connecting the points \( \pm 2 \). In the illustrations provided here, Figs. 7 and 8, we have used a rather elongated ellipse corresponding to the parameter choice \( a = 0.7 \).

Fig. 7 provides an example from the very simplest problem in this category: the result of particle advection in the flow field due to a freely moving elliptical cylinder. In this calculation, the mass of the ellipse is equal to the mass of the fluid it displaces—had gravity been included, the elliptical cylinder would be neutrally buoyant. The cylinder has no circulation about it and in this case no vortices have been introduced into the flow field. The entire fluid motion is due to the moving body. The body motion is integrable as first shown by Kirchhoff, cf. [35]. However, the motion of the particles around it is apparently chaotic as illustrated in Fig. 7. Two particles were started virtually at the same location: the deviation in initial positions is \( 10^{-3} \) times the dimension of the ellipse. Hence, the two particles initially follow virtually the same path. At some point, however, the exponential deviation in their long-time trajectories becomes apparent, and the two particles move to opposite sides of the ellipse! This is another example of the chaotic advection phenomenon from Section 2.

Returning to another theme of Section 2 we may ask: Is it possible for a rigid body, translating and rotating in an ideal fluid, to carry an atmosphere with it in its motion? Clearly, the uniformly translating circular cylinder analyzed by Maxwell [6] does not carry fluid with it. Had we introduced a circulation around the cylinder, on the other hand, it could have an atmosphere, as mentioned in Section 2. But what if we consider an ellipse, with no circulation about it, that is spinning and translating integrably as per Kirchhoff’s solution? In the limit of pure rotation Morton [11] showed that there would be “islands” on either side of the ellipse. As Lamb [35] wrote, when reviewing this work: “The paths followed by the particles of fluid in several...cases, as distinguished from the streamlines, have been studied by Prof. Morton; they are very remarkable”. The result lay dormant for four decades until Darwin rediscovered it [12], apparently without being aware of the work by Morton [11]. We may add a modern twist to this result: if the islands are slightly perturbed by adding a small translation to the body motion, KAM theory shows that some kind of islands remain. Roenby and Aref [34] show that for a predominantly spinning motion of the ellipse, it may indeed carry an atmosphere with it. The atmosphere breaks up into regular “islands” embedded in a chaotic “sea”. Numerical experiments [34] show that so long as the ratio of translational velocity to angular velocity multiplied by a characteristic, linear dimension of the body is 0.1 or less, a discernible atmosphere is present. In Fig. 7 we have exploited our knowledge of where the chaos is within such an atmosphere to pick initial positions that are very close in the chaotic part of the atmosphere. The two particle trajectories follow one another for a long time (since the initial conditions are so close) but ultimately deviate considerably due to exponential separation of close orbits in a chaotic system.

We have also studied the case of an ellipse and one vortex, when the circulation around the body is opposite to the circulation of the vortex [33]. This corresponds physically to the case when the vortex has been shed from the body by viscous flow processes that are, of course, not captured in the model. This is certainly an important special case from the point of view of applications.

Fig. 8 shows two examples of trajectories of the center of an ellipse and of a point vortex with which it interacts. In these examples the mass of the ellipse equals one quarter the mass of displaced fluid. Both examples have the same values of linear and angular impulse and of the Hamiltonian (kinetic energy) of the motion. The differences in the trajectories come from a small change in the initial position of the vortex that was constructed so that it preserves the constants of motion. Visual observation suggests that the top trajectories in Fig. 8 show a predominantly regular motion of both body and vortex, while the bottom trajectories show chaotic motion of both. These conclusions are borne out by considering other measures, in particular appropriately constructed Poincaré sections of the motion [33]. One curious observation is that when regular and irregular motions exist side by side for the same values of the integrals (and for all system parameters), the irregular motion makes greater headway in a fixed interval of time than the regular motion. This is clear from Fig. 8 where both interaction sequences have been traced for the same time interval.

There is a gross similarity between the body–vortex motions in Fig. 8 and the forward motion of a vortex pair, Fig. 3(c). The vortex and the body, which has a circulation around it that is opposite to the circulation of the vortex, move forward together. However, the strong body–vortex interactions arising from having the body and the vortex close to one another induce the many nonlinear excursions that are evident. For some initial conditions the paths of both vortex and body center are predominantly regular, for others they are irregular. This is similar to the Lagrangian chaos seen in the vortex interaction problems discussed in Section 3.

5. Concluding remarks

In conclusion we make three points. First, consider 2D inviscid hydrodynamics written in the Lagrangian representation as a pair of PDEs for the position coordinates of a fluid particle (PDEs because the Lagrangian positions depend on the initial coordinates and on time). Our examples suggest that this is unlikely to be a useful approach in general. We have seen that already within the “universe” of a rigid body, a few point vortices and a passively advected particle in individual material points in the same flow can have either integrable (regular) or chaotic (irregular) motions. For example, three (and in special cases four) interacting vortices will (can) have regular trajectories while a particle they advect will move irregularly (more or less so depending on where it is started in the flow). Similarly, a rigid body may move regularly in response to fluid reactive forces, but the flow field it induces by its motion advects particles chaotically. Encompassing both these types of motion in a single solution of the aforementioned PDEs is not possible. In terms of describing the flow field, then, the conventional Eulerian representation is the preferred approach since the additional level of detail given by individual particle trajectories is “hidden”. Abrashkin and Yakubovich [36,37] attempted to study 2D ideal flow with embedded vorticity using the Lagrangian representation. However, they had to confine
Fig. 7. Trajectories of two particles (large amplitude black curves) advected by the motion of a homogeneous elliptical cylinder in an ideal fluid. The center of the ellipse moves left-to-right along the small-amplitude, wiggly curve. Initial and final positions of the cylinder are shown by grey outlines. Initial positions (open circles) of the two particles are virtually identical. Initially the particles follow the same path. After some time, however, one particle moves to the other side of the ellipse. Final positions of particles are shown as black dots. The motion of the cylinder is integrable, the motion of the particles chaotic. For additional details see [34].

Fig. 8. Trajectories of a point vortex (black curve) and the center of a homogeneous elliptical cylinder (grey curve), with a circulation around it that is opposite to the vortex, as they interact in an ideal fluid. The cylinder has one quarter of the mass of the fluid it displaces. The initial and final positions of the ellipse are shown. The initial and final positions of the point vortex are shown as open circles. Both trajectory segments are for the same length of time. Top: the body–vortex interaction is predominantly regular. Bottom: chaotic body–vortex interaction. The initial positions of the vortices in the two sequences are only slightly different. For additional details see [33].

Fig. 9. Trajectories of three vortices (black curves) in a square, doubly-periodic domain during one period of the relative motion. This problem is integrable and periodic, yet the trajectories are quite complex. The inset shows a close-up of the basic square (dashed lines). Vortices are labeled according to their relative circulations, 7, 3 and −10. Initial (solid circles) and final (open circles) vortex positions are joined (grey curves) to highlight the periodicity of the relative motion. The vortices travel well beyond the boundaries of the basic square and interact with numerous periodic images, which are not shown. Source: From [38].

themselves to solutions with simple time dependence of the vorticity field. A more complicated vorticity field would immediately bring in chaotic motions. We suggest that our simple examples, even though they involve singular point vortices, illustrate why a purely Lagrangian representation runs into difficulties.

Second, we point out that one cannot, in all cases, decide whether particle motions are regular or chaotic simply by looking at them. We have been somewhat cavalier with the assignment of labels such as “integrable” and “chaotic”, or “regular” and “irregular”, in the preceding developments. However, there are more rigorous definitions and arguments behind these assignments, and further numerical diagnostics in the chaotic/irregular case, that support the assignments we have made. Nevertheless, the issue arises if one can simply “see” integrability versus chaos “with the naked eye” from the regularity or irregularity of particle trajectories. From the examples given up to now it would appear that one can.

We provide a cautious note with Fig. 9 taken from calculations of integrable motion of three vortices in a square domain with periodic boundary conditions [38]. Although the interactions are more complex than in Eq. (3.1)—the periodic images result in the inter-vortex interactions involving Weierstrass elliptic functions—the symplectic structure is similar, and the three-vortex problem is integrable when the sum of the vortex circulations vanishes. See [38] for details. Similar, although less dramatic, examples are seen in the integrable case of three vortices, with net circulation zero, in a periodic strip [39]. The example shown is from the case of circulations 7, 3, and −10 (only the ratio of circulations matters).

For rational ratios between the vortex strengths, the relative motion is periodic except for separatrices. Thus, in principle, the motion is quite simple. What comes as a surprise is that the motion during a period can be very complex. Visual observation of the three trajectories in Fig. 9, without knowing that the relative motion is periodic, would not immediately suggest integrability! Indeed, if the ratios between the vortex circulations are irrational, the problem is still Liouville integrable but the relative motions are not periodic. In terms of complexity and convolution these trajectories compare well to computed trajectories of strong isolated vortices in numerical simulations of 2D Navier–Stokes turbulence, even though a vortex in the turbulence simulation may encounter additional processes such as merging or tearing. One point of Fig. 9, then, is to underscore that visual observation of trajectories alone may be ambiguous in assessing integrability versus chaos. Closer analysis of the particle trajectories is required, and any assessment should be corroborated by looking at further diagnostics.

Finally, we remark that chaos in the Lagrangian representation of a flow may seem a rather subtle thing. As we said previously, it is largely invisible if one works in the Eulerian representation. However, the chaotic motion of individual particles or bodies provides a key mechanism in a number of situations. Apart from the well studied applications to stirring and mixing of viscous fluids, we submit that chaotic vortex motion and chaotic interactions between vortices and bodies have potential applications to areas such as turbulence, flow-structure interactions, and advective transport. In the Eulerian framework it would be difficult to distinguish three vortices moving regularly while advecting particles chaotically. Similarly, it would be difficult to distinguish chaotic and regular body-vortex interactions. These issues are intrinsically Lagrangian both in their formulation and elucidation. The fluctuations in particle positions, or in forces on a body, are quite different in the regular and chaotic regimes. In order to understand such differences, and their consequences, one has to adopt a Lagrangian description.

Acknowledgements

We thank members of the Center for Fluid Dynamics at DTU for comments and discussion. This work is supported by a Niels Bohr Visiting Professorship at the Technical University of Denmark sponsored by the Danish National Research Foundation.

References

[9] M.A. Stremler, J.E. Marsden, The Hamiltonian structure of a 2D rigid circular cylinder interacting dynamically with point vortices, Phys. Fluids 14 (2002) 1214–1227; [38] for details. Similar, although less dramatic, examples are seen in the integrable case of three vortices, with net circulation zero, in a periodic strip [39]. The example shown is from the case of circulations 7, 3, and −10 (only the ratio of circulations matters).


Please cite this article in press as: H. Aref, et al., Nonlinear excursions of particles in ideal 2D flows, Physica D (2010), doi:10.1016/j.physd.2010.08.007